

Exercise Sheet #4

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- P1.** Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f : X \rightarrow \mathbb{C}$ be a measurable function. Then prove that $f = 0$ a.e. if and only if $\int_E f d\mu = 0$ for every measurable set $E \in \mathcal{B}$.
- P2.** Prove that a space is complete if and only if the following property holds: for functions $f, g; X \rightarrow \mathbb{C}$, if f is measurable and $f = g$ a.e., then g is also measurable.
- P3.** Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{B} : \mu(N) = 0\}$ be the σ -ideal of μ -null sets. Show that the family $\overline{\mathcal{B}} = \{E \cup F : E \in \mathcal{B}, F \subseteq N \in \mathcal{N}\}$ is a σ -algebra, and there is a unique extension $\bar{\mu}$ of μ to $\overline{\mathcal{B}}$.
- P4.** Let $(X, \mathcal{B}, \mu)$ be a measure space. Prove that simple functions are dense in $L^1(X)$.
- P5.** Let $(X, \mathcal{B}, \mu)$ be a measure space and $E \in \mathcal{B}$ a set of positive measure. If $\{E_n\}_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets and $E = \bigcup_{n \in \mathbb{N}} E_n$ prove that for every function $f \in L^1(X)$

$$\lim_{n \rightarrow \infty} \int_{E_n} f d\mu = \int_E f d\mu$$

and state and prove an analogous result for decreasing sequences.

- P6.** Prove (yet again) the identity

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k \in \mathbb{N}_0} \frac{x^k}{k!}.$$

Hint: Consider the measure space $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}), \nu)$ where ν is the counting measure on $\mathbb{N}_0$ ($\nu(A) = |A|$ for each $A \subseteq \mathbb{N}_0$). In addition, for each $n \in \mathbb{N}$ define

$$f_n(k) = \begin{cases} \binom{n}{k} \frac{x^k}{n^k} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}.$$